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Explicit error bounds for multivariate analytic combinatorics

Joint work with Stephen Melczer (smelczer@uwaterloo.ca)

AMS Spring Sectional Meeting 2026, North Dakota State University, Fargo, ND

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Can one compute asymptotics for the *diagonal coefficient sequence* $(f_{n\boldsymbol{r}})_{n=0}^{\infty}$?

Applications

Diagonals of rational functions appear in the study of...

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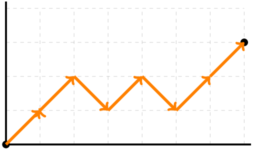

Lattice paths

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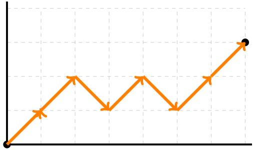

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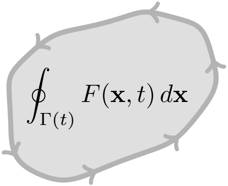

Period integrals

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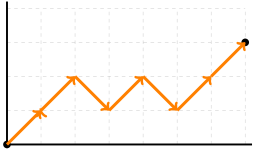

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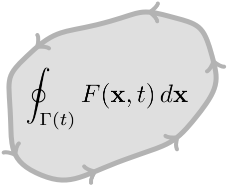

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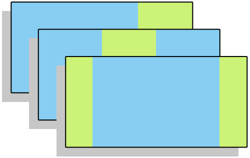

Irrational tilings

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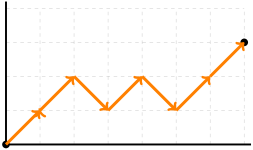

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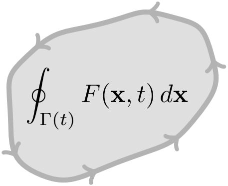

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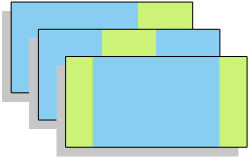

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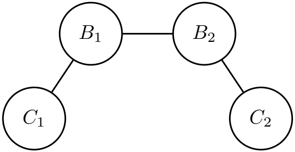

Graphs and Networks

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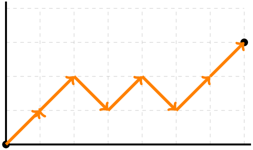

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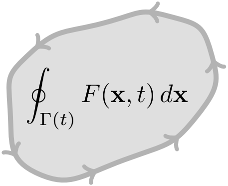
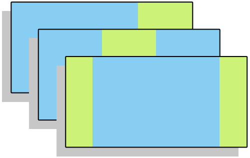


Period integrals

...AND MUCH MORE!



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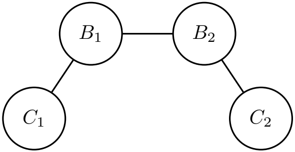

Graphs and Networks

Another Problem: Positivity

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- Ibrahim, *Positivity Proofs for Linear Recurrences with Several Dominant Eigenvalues*, 2025

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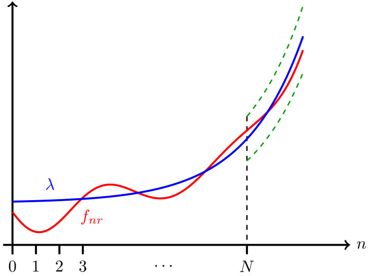

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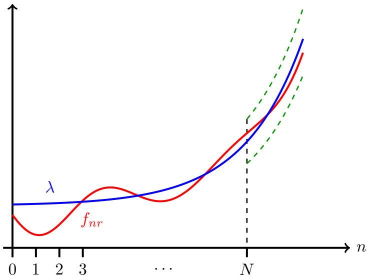

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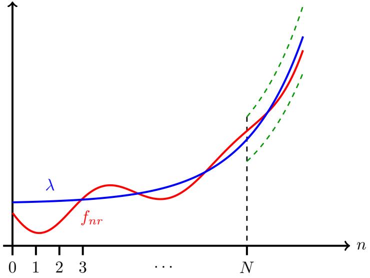

This motivates the need for explicit error bounds!

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Software:

```
[1]: from ore_algebra import OreAlgebra
     from ore_algebra.analytic.singularity_analysis import bound_coefficients
     Pol.<z> = PolynomialRing(QQ)
     Dop.<Dz> = OreAlgebra(Pol) # Dz represents the operator d/dz
     dop = (z^2*(4*z - 1)*(4*z + 1)*Dz^3 + 2*z*(4*z+1)*(16*z-3)*Dz^2 + 2*(112*z^2 + 14*z - 3)*Dz + 4*(16*z + 3))
     bound_coefficients(dop, [1, 2, 6], order=2)

[1]: 1.000000000000000*4^n*(([1.27323954473516 +/- 3.09e-15] + [+/- 2.57e-25]*I)*n^(-1) + ([-1.90985931710274 +/- 4.63e-15] + [+/- 3.86e-25]*I)*n^
     (-2) + B([1503.374850517149 +/- 3.85e-13]*n^(-3)*log(n), n >= 7))
```

See `bound_coefficients` in `Ore_Algebra` package, or [DMM 2023].

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- How do we compute $\lambda(n)$ and $N \in \mathbb{N}$ such that

$$|f_{n\boldsymbol{r}} - \lambda(n)| \leq \frac{1}{2}|\lambda(n)|$$

for all $n \geq N$?

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- **Step 1:** Start with Cauchy integral expression

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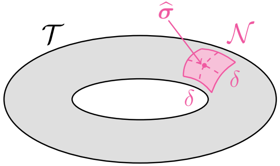
- **Step 5:** *Localize* Cauchy integral to nbd $\mathcal{N}$ of $\boldsymbol{\sigma}$.



ACSV

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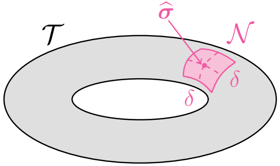

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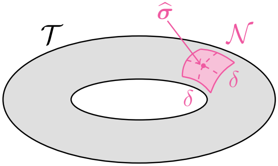

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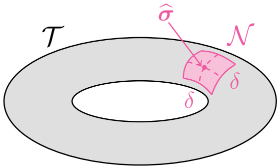

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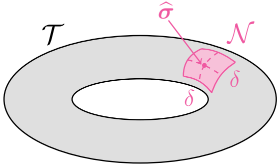

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Our Results

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Let F, $\boldsymbol{r}$, and χ be as above, and suppose $\boldsymbol{\sigma}$ is strictly minimal smooth critical. Then one can construct a $c > 0$ and $0 < \tau < |\boldsymbol{\sigma}|^{-\boldsymbol{r}}$ so that for all n,

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Proposition 2

Let $\boldsymbol{\sigma}$, χ be as above with $\boldsymbol{\sigma}$ nondegenerate. Let $\alpha \in \mathbb{Q} \cap (1/3, 1/2)$ and define

$$\lambda(n) := \frac{\boldsymbol{\sigma}^{-n\boldsymbol{r}}}{n^{(d-1)/2}} \frac{(2\pi)^{(d-1)/2} A(\mathbf{0})}{\sqrt{\det \mathcal{H}}}.$$

Then one can construct $N \in \mathbb{N}$ and $C > 0$ such that

$$|\chi - \lambda| \leq C|\lambda| n^{1-3\alpha}.$$

for all $n \geq N$.

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Again, one can extend this argument to the finitely minimal case without much difficulty.

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$$F(x, y) = \frac{1}{1 - ax - by + cxy} \quad a, b, c \geq 0, \ ab > c.$$

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Gillis-Reznick-Zeilberger

$$F(z_1, ..., z_d) = \frac{1}{1 - \sum_i z_i + d! \prod_i z_i}, \qquad d \geq 4.$$

With $d = 4$ we get $\boldsymbol{N > 10^6}$.

Aside / Advertisement

We wrote a Sage package for computing diagonals...

```
[5]: from sage_periods import *
     var( 'x y z w')
     F = 1/(1-x-y-z-w + 24*x*y*z*w)

     %time L = compute_diagonal_annihilator(F)
     L

     CPU times: user 896 ms, sys: 3.09 ms, total: 899 ms
     Wall time: 897 ms

[5]: (t^8 + 5/36*t^7 + 31/5184*t^6 - 23/31104*t^5 + 55/2985984*t^4 - 19/107495424*t^3 + 1/1719926784*t^2)*Dt^3 + (6*t^7 + 13/24*t^6 + 1/192*t^5 -
     37/20736*t^4 + 13/248832*t^3 - 13/23887872*t^2 + 1/573308928*t)*Dt^2 + (7*t^6 + 5/18*t^5 - 23/1728*t^4 - 1/6912*t^3 + 11/995328*t^2 - 7/3583
     1808*t + 1/1719926784)*Dt + t^5 - 1/72*t^4 - 1/864*t^3 + 1/41472*t^2 - 1/995328*t

[11]: var('z')
      F2 = 1/(1-x-y)
      %time L2 = compute_diagonal_annihilator(F2,r=[2,3],vari=[x,y],t=z, Dt = "D_z")
      show(L2)

      CPU times: user 350 ms, sys: 3.12 ms, total: 353 ms
      Wall time: 353 ms
```

$$\left(z^4 - \frac{108}{3125}z^3\right)D_z^4 + \left(8z^3 - \frac{486}{3125}z^2\right)D_z^3 + \left(\frac{72}{5}z^2 - \frac{348}{3125}z\right)D_z^2 + \left(\frac{24}{5}z - \frac{12}{3125}\right)D_z + \frac{24}{625}$$

```
[14]: diag = fast_diagonal_series(F2,z,10,[2,3])
      diag

[14]: 1 + 10*z + 210*z^2 + 5005*z^3 + 125970*z^4 + 3268760*z^5 + 86493225*z^6 + 2319959400*z^7 + 62852101650*z^8 + 1715884494940*z^9 + 47129212243
      960*z^10 + O(z^11)

[15]: L2(diag)

[15]: O(z^10)
```

https://github.com/ACSVMath/sage_periods

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Thank you! :)